A Comparison of Nodal- and Mesh-Based Magnetic Equivalent Circuit Models

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Outline

• Magnetic Equivalent Circuit (MEC) Modeling

• Alternative MEC Formulations
  – Nodal-based
  – Mesh-based

• Comparison of Numerical Properties
MEC Model of Claw-Pole Machine
MEC Sources

- Magnetomotive Force
  - Result of Ampere’s current Law
  - Represents effects of winding currents
  - Incorporates winding layout
  - Similar to a voltage source
MEC Flux Tubes

- Flux Tubes
  - Shape determined by engineering judgment
  - Establish topology of MEC network
  - Incorporate geometry of the machine
Node Potentials and Reluctance

- Magnetic Scalar Potentials
  - Represent node potentials

- Reluctance
  - Calculated from geometry of flux tube
  - Similar to resistance
  - Allows for effects of magnetic saturation

\[ R = \int \frac{dx}{\mu(x)A(x)} \]
Example MEC-Based Design Program

% Stator Input Data

%------------------------------------------------------------------------------
% Stator Input Data
%------------------------------------------------------------------------------
OD = 0.12985;  % STATOR OUTER DIAMETER, m
ID = 0.09662;  % STATOR INNER DIAMETER, m
GLS = 26.97e-3;  % STATOR STACK LENGTH, m
DBS = 4.98e-3;  % STATOR YOKE DEPTH, m
SFL = 0.99;  % STACKING FACTOR
H0 = 0.64e-3;  % STATOR SLOT DIMENSION, m
H1 = 0.0;  % STATOR SLOT DIMENSION, m
H2 = 1.3e-3;  % STATOR SLOT DIMENSION, m
B0 = 2.4617e-3;  % STATOR SLOT DIMENSION, m
SYN = 2.4e-3;  % STATOR YOKE NOTCH RADIUS, (weight calculation only),
SLTINS = 2.997e-4;  % SLOT INSULATION WIDTH, m
G1 = 0.305e-3;  % MAIN AIR GAP LENGTH, m
SAWG = 13.75;  % WIRE GUAGE OF ARMATURE WINDING, 1.29e-3
TC = 11.0;  % NUMBER OF TURNS PER COIL
ESC = 2.54e-3;  % ARMATURE WINDING EXTENSION BEYOND STACK
RSC = 7.62e-3;  % ARMATURE WINDING RADIUS BEYOND STACK
CPIT = 3.0;  % COIL PITCH IN TEETH
STW = 3.86e-3;  % WIDTH OF TOOTH SHANK, m
DENS = 7872.0;  % DENSITY OF IRON, ROTOR & STATOR, kg/m^3
SLTH = 0.828e-3;  % STATOR LAMINATION THICKNESS, m

% Rotor Input Data

%------------------------------------------------------------------------------
% Rotor Input Data
%------------------------------------------------------------------------------
TED = 12.0e-3;  % ROTOR END DISK THICKNESS, m
DC = 50.0e-3;  % ROTOR CORE DIAMETER, m
CL = 28.1e-3;  % ROTOR CORE LENGTH, m
GLP = 27.0e-3;  % LENGTH OF ROTOR POLE, m
CID = 51.5e-3;  % FIELD COIL INNER DIAMETER, m
COD = 74.0e-3;  % FIELD COIL PLASTIC SLOT OUTER DIAMETER, m
COILW = 28.0e-3;  % FIELD COIL WIDTH, m
WPT = 7.39e-3;  % ROTOR TOOTH WIDTH AT TIP OF TOOTH, arclength, m
WPR = 27.0e-3;  % ROTOR TOOTH WIDTH AT ROOT OF TOOTH, arclength, m
HPT = 2.997e-3;  % ROTOR TOOTH HEIGHT AT TIP OF TOOTH, m
HPR = 11.38e-3;  % ROTOR TOOTH HEIGHT AT ROOT OF TOOTH, m
TRAD = 1.19e-3;  % ROTOR TOOTH BEND RADIUS, m
RP = 12.0;  % NUMBER OF POLES
TRC = 306.0;  % FIELD WINDING NUMBER OF TURNS
RAWG = 19;  % WIRE GUAGE OF FIELD WINDING, 0.813e-3
TFLD = 48.8;  % HOT FIELD TEMPERATURE, C
TA = 20.0;  % AMBIENT TEMPERATURE, C
SD = 17.575e-3;  % SHAFT DIAMETER, m
DD = 58.6e-3;  % DISK DIAMETER (claw V to claw V), m
CW = 7.897e-2;  % CHAMFER WIDTH, rad
CD = 3.0*G1;  % CHAMFER DEPTH, m
## Summary Table

<table>
<thead>
<tr>
<th>SHAFT TORQUE</th>
<th>Avg Torque: 7.369e+000 [N(\cdot)m]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ripple Torque: 3.148e+000 [N(\cdot)m/rev]</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>PEAK MAGNETIC FLUX DENSITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Claw Section 1 (tip): 1.140e+000 [T]</td>
</tr>
<tr>
<td>Claw Section 3: 1.455e+000 [T]</td>
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<tr>
<td>Claw Section 5: 1.600e+000 [T]</td>
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<tr>
<td>Claw Section 7: 1.738e+000 [T]</td>
</tr>
<tr>
<td>Yoke Piece 1: 1.260e+000 [T]</td>
</tr>
<tr>
<td>Tooth 1: 1.664e+000 [T]</td>
</tr>
<tr>
<td>End Piece: 1.336e+000 [T]</td>
</tr>
<tr>
<td>End Disk Piece: 1.378e+000 [T]</td>
</tr>
<tr>
<td>Shaft: 1.333e+000 [T]</td>
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<tr>
<td>Core: 1.483e+000 [T]</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>PEAK AMPERE-TURNS</th>
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<tbody>
<tr>
<td>Claw Section 1 (tip): 5.865e+000 [A-turn]</td>
</tr>
<tr>
<td>Claw Section 3: 2.027e+001 [A-turn]</td>
</tr>
<tr>
<td>Claw Section 5: 3.870e+001 [A-turn]</td>
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<tr>
<td>Claw Section 7: 9.008e+001 [A-turn]</td>
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<tr>
<td>Yoke Piece 1: 7.179e+000 [A-turn]</td>
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<tr>
<td>Tooth 1: 5.349e+001 [A-turn]</td>
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<tr>
<td>End Piece: 3.777e+001 [A-turn]</td>
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<tr>
<td>End Disk Piece: 7.190e+001 [A-turn]</td>
</tr>
<tr>
<td>Shaft: 1.040e+002 [A-turn]</td>
</tr>
<tr>
<td>Core: 1.040e+002 [A-turn]</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>INDUCTANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lasid (aag fundamental): 7.270e-003 [H]</td>
</tr>
<tr>
<td>Lasas (average): 4.642e-004 [H]</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>MISC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radial Force: 1.115e+002 [N]</td>
</tr>
<tr>
<td>RMS EMF: 2.922e+001 [V]</td>
</tr>
<tr>
<td>Stator Slot Fill: 6.032e+001 [%]</td>
</tr>
<tr>
<td>Rotor Slot Fill: 7.207e+001 [%]</td>
</tr>
<tr>
<td>Stator Resistance: 8.323e-002 [ohm]</td>
</tr>
<tr>
<td>Field Resistance: 1.761e+000 [ohm]</td>
</tr>
<tr>
<td>Field Hot Temp: 4.880e+001 [°C]</td>
</tr>
</tbody>
</table>

### STATOR VARIABLES
- **Phase Current:** 2.256e+001 [Arms] 3.751e+001 [A op]
- **Phase Voltage:** 1.768e+001 [Vrms] 2.442e+001 [V op]

### DC VARIABLES
- **Output DC Current:** 4.995e+001 [A avg] 1.698e+001 [A ripple]
- **Output DC Voltage:** 1.790e+001 [V avg] 5.841e+000 [V ripple]

### FIELD VARIABLES
- **Field Current:** 4.546e+000 [A avg] 5.164e+001 [A ripple]
- **Field Voltage:** 8.094e+000 [V]

### WEIGHTS
- **Teeth Weight:** 3.550e+001 [kg]
- **Yoke Piece Weight:** 4.148e+001 [kg]
- **Total Stator Weight:** 7.698e+001 [kg]
- **Rotor Segment Weight:** 1.336e+000 [kg]
- **Rotor Core Weight:** 3.780e+001 [kg]
- **Permanent Magnet Weight:** 0.000e+000 [kg]
- **Total Rotor Weight:** 1.714e+001 [kg]
- **Total Magnetic Weight:** 2.484e+001 [kg]
- **Stator Winding Weight:** 5.306e-001 [kg]
- **Rotor Winding Weight:** 3.655e-001 [kg]
- **Total Copper Weight:** 8.760e-001 [kg]
- **Total Weight:** 3.360e+000 [kg]

### EFFICIENCY & LOSS
- **Efficiency:** 6.172e+001 [%]
- **Avg Power In:** 1.465e+003 [W]
- **Avg Power Out:** 9.045e+002 [W]
- **Friction & Windage Loss:** 1.984e+001 [W]
- **Stator Core Loss. Fund:** 9.080e+001 [W]
- **Stator Core Loss. 3*Fund:** 1.980e+000 [W]
- **Stator Core Loss. 5*Fund:** 5.414e+000 [W]
- **Stator Core Loss. 7*Fund:** 4.557e-001 [W]
- **Rotor Pole Face Loss:** 7.993e+001 [W]
- **Stator I^2 R Loss (3-PH):** 1.274e+002 [W]
- **Diode Losses (6):** 2.003e+002 [W]
- **Field I^2 R Loss:** 3.719e+001 [W]
Example System for Research
Presented Herein
Nodal Analysis of Example

\[ \mathbf{A}_p \mathbf{u} = \varphi \]

\[ \mathbf{u} = \begin{bmatrix} u_{m1} & u_{c11} & u_{c12} & u_{c13} & u_{s1} & u_{s2} & u_{c23} & u_{c22} & u_{c21} & u_{m2} \end{bmatrix}^T \]

\[ \varphi = \begin{bmatrix} F_m P_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -F_m P_m \end{bmatrix}^T \]
Nodal-Based Matrices

\[ \mathbf{A}_P = \begin{bmatrix} \mathbf{A}_{P1} & \mathbf{A}_{P2} \\ \mathbf{A}_{P2}^T & \mathbf{A}_{P4} \end{bmatrix} \quad \mathbf{A}_{P1} = \begin{bmatrix} p_m + p_{c1} & -p_{c1} & 0 & 0 & 0 \\ -p_{c1} & p_{ag} + p_{c1} + p_{c2} & -p_{c2} & 0 & -p_{ag} \\ 0 & -p_{c2} & p_{ag} + p_{c2} + p_{c3} & -p_{c3} & -p_{ag} \\ 0 & 0 & -p_{c3} & p_{ag} + p_{c3} & -p_{ag} \\ 0 & -p_{ag} & -p_{ag} & -p_{ag} & 3p_{ag} + p_s \end{bmatrix} \]

\[ \mathbf{A}_{P2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -p_s & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}_{P4} = \begin{bmatrix} 3p_{ag} + p_s & -p_{ag} & -p_{ag} & -p_{ag} & 0 \\ -p_{ag} & p_{ag} + p_{c3} & -p_{c3} & 0 & 0 \\ -p_{ag} & -p_{c3} & p_{ag} + p_{c2} + p_{c3} & -p_{c2} & 0 \\ -p_{ag} & 0 & -p_{c2} & p_{ag} + p_{c1} + p_{c2} & -p_{c1} \\ 0 & 0 & 0 & -p_{c1} & p_m + p_{c1} \end{bmatrix} \]
Mesh Analysis of Example System

\[ \mathbf{A}_R \mathbf{\varphi} = \mathbf{F} \]

\[ \mathbf{\varphi} = [\phi_m \ \phi_{c11} \ \phi_{c12} \ \phi_{c22} \ \phi_{c21}]^T \]

\[ \mathbf{F} = [2F_m \ \ 0 \ \ 0 \ \ 0 \ \ 0]^T \]
Mesh-Based Matrix

\[
A_R = \begin{bmatrix}
R_M & -R_{c12} & -R_{c13} - R_{ag} & R_{c23} + R_{ag} & R_{c22} \\
-R_{c12} & R_{c12} + 2R_{ag} & -R_{ag} & 0 & 0 \\
-R_{c13} - R_{ag} & -R_{ag} & R_{c13} + 2R_{ag} & 0 & 0 \\
R_{c23} + R_{ag} & 0 & 0 & R_{c23} + 2R_{ag} & -R_{ag} \\
R_{c22} & 0 & 0 & -R_{ag} & R_{c22} + 2R_{ag}
\end{bmatrix}
\]
Modeling Magnetic Materials

Nonlinear – Saturation of Material

$B-H$ Curve

$\mu - B$ Curve
Solving Nodal Formulation

\[ f(u) = A_P u - \phi = 0 \]

Utilizing Newton Raphson

\[ u_{n+1} = u_n - \left[ J(u_n) \right]^{-1} f(u_n) \]

\[ f_1(u) = (P_m + P_{c1})u_{m1} - P_{c1}u_{c11} - F_mP_m = 0 \]

\[ \Phi_{c1} = P_{c1}(u_{m1} - u_{c11}) \quad B_{c1} = \frac{\Phi_{c1}}{A_{c1}} \]

\[ J_{11} = \frac{\partial f_1}{\partial u_{m1}} = (P_m + P_{c1}) + \frac{\partial P_{c1}}{\partial u_{m1}}(u_{m1} - u_{c11}) \]
Closed-form Expression for Jacobian

\[
\frac{\partial P_{cl}}{\partial u_{ml}} = \frac{\partial P_{cl}}{\partial \mu_{rel}} \frac{\partial \mu_{rel}}{\partial B_{cl}} \frac{\partial B_{cl}}{\partial \Phi_{cl}} \frac{\partial \Phi_{cl}}{\partial u_{ml}}
\]

\[
\frac{\partial P_{cl}}{\partial \mu_{rel}} = \mu_0 N \left( l_1 - l_2 \right) = \frac{P_{cl}}{\mu_{rel}}
\]

\[
\frac{\partial \mu_{rel}}{\partial B_{cl}} = \lambda_{caw} \ln \left( \frac{l_1}{l_2} \right)
\]

\[
\frac{\partial B_{cl}}{\partial \Phi_{cl}} = \frac{1}{A_{cl}}
\]

\[
\frac{\partial \Phi_{cl}}{\partial u_{ml}} = \frac{\partial P_{cl}}{\partial u_{ml}} (u_{ml} - u_{c11}) + P_{cl}
\]

So, one can establish closed-form expression

\[
\frac{\partial P_{cl}}{\partial u_{ml}} = \frac{\alpha \beta \gamma P_{cl}}{1 - \alpha \beta \gamma (u_{ml} - u_{c11})}
\]
Solving Mesh Formulation

\[ g(\phi) = A_R \phi - F = 0 \]

\[ g_1(\phi) = R_M \phi_m - R_{c12} \phi_{c11} - \left( R_{c13} + R_{ag} \right) \phi_{c12} \]

\[ + \left( R_{c23} + R_{ag} \right) \phi_{c22} + R_{c22} \phi_{c21} - 2F_m = 0 \]

\( R_{c11}, R_{c12}, R_{c13}, R_{c21}, R_{c22}, \) and \( R_{c23} \) are flux-dependent reluctances

\[ J_{11} = R_M + \frac{\partial R_{c11}}{\partial \phi_m} \phi_m + \frac{\partial R_{c21}}{\partial \phi_m} \phi_m + \frac{\partial R_{c12}}{\partial \phi_m} \left( \phi_m - \phi_{c11} \right) + \]

\[ \frac{\partial R_{c22}}{\partial \phi_m} \left( \phi_m + \phi_{c21} \right) + \frac{\partial R_{c13}}{\partial \phi_m} \left( \phi_m - \phi_{c12} \right) + \frac{\partial R_{c23}}{\partial \phi_m} \left( \phi_m + \phi_{c22} \right) \]
Closed-form Expression for Jacobian

\[
\frac{\partial R_{c_{12}}}{\partial \phi_m} = \frac{\partial R_{c_{12}}}{\partial \mu_{rc_{12}}} \frac{\partial \mu_{rc_{12}}}{\partial B_{c_{12}}} \frac{\partial B_{c_{12}}}{\partial \Phi_{c_{12}}} \frac{\partial \Phi_{c_{12}}}{\partial \phi_m} \quad \Rightarrow \quad \frac{\partial R_{c_{12}}}{\partial \mu_{rc_{12}}} = -\frac{1}{\mu_{rc_{12}}} R_{c_{12}}
\]

\[
\frac{\partial \mu_{rc_{12}}}{\partial B_{c_{12}}} \quad \text{Obtained from } \mu-B \text{ curve}
\]

\[
\frac{\partial B_{c_{12}}}{\partial \Phi_{c_{12}}} = \frac{1}{A_{c_{12}}}
\]

\[
\frac{\partial \Phi_{c_{12}}}{\partial \phi_m} = -1
\]

So, one can establish closed-form expression (much less tedious than permeance-based)

\[
\frac{\partial R_{c_{12}}}{\partial \phi_m} = \frac{R_{c_{12}}}{\mu_{rc_{12}} A_{c_{12}}} \frac{\partial \mu_{rc_{12}}}{\partial B_{c_{12}}}
\]
Comparison of Nodal and Mesh-Based Formulations

\[ H_c = 10^5 \text{ A/m} \]

- Nodal Model Approx. Jacobian
- Nodal Model Analytic Jacobian
- Mesh Model Analytic Jacobian

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodal Model Approx. Jacobian</td>
<td>3 iterations</td>
</tr>
<tr>
<td>Nodal Model Analytic Jacobian</td>
<td>3 iterations</td>
</tr>
<tr>
<td>Mesh Model Analytic Jacobian</td>
<td>4 iterations</td>
</tr>
</tbody>
</table>

Air Gap Flux Density (T)

\( x \) (mm)
Comparison of Nodal and Mesh-Based Formulations

\[ H_c = 8 \times 10^5 \text{ A/m} \]

- \( \times \) Nodal Model Approx. Jacobian
- \( \triangle \) Mesh Model Analytic Jacobian

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nodal Model Approx. Jacobian</td>
<td>5 iterations</td>
</tr>
<tr>
<td>Nodal Model Analytic Jacobian</td>
<td>DNC</td>
</tr>
<tr>
<td>Mesh Model Analytic Jacobian</td>
<td>5 iterations</td>
</tr>
</tbody>
</table>
Comparison of Nodal and Mesh-Based Formulations

\[ H_c = 16 \times 10^5 \text{ A/m} \]

- x: Nodal Model Approx. Jacobian
- ▲: Mesh Model Analytic Jacobian

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Convergence</th>
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</thead>
<tbody>
<tr>
<td>Nodal Model Approx. Jacobian</td>
<td>53 iterations</td>
</tr>
<tr>
<td>Nodal Model Analytic Jacobian</td>
<td>DNC</td>
</tr>
<tr>
<td>Mesh Model Analytic Jacobian</td>
<td>6 iterations</td>
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</table>
Interpretation of Results

\[ x_{n+1} = x_n - [J(x_n)]^{-1} f(x_n) \]

<table>
<thead>
<tr>
<th>Implementation</th>
<th>Condition Number</th>
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<tbody>
<tr>
<td>Nodal Model Approx. Jacobian</td>
<td>~ 2 x 10^5</td>
</tr>
<tr>
<td>Nodal Model Analytic Jacobian</td>
<td>~ 2 x 10^8</td>
</tr>
<tr>
<td>Mesh Model Analytic Jacobian</td>
<td>~ 500</td>
</tr>
</tbody>
</table>

Ill-conditioning mainly due to difference in airgap permeance and claw permeances
Scaling to Help?

\[
A_{p_1} = \begin{bmatrix}
P_m + P_{c_1} & -P_{c_1} & 0 & 0 & 0 \\
-P_{c_1} & P_{ag} + P_{c_1} + P_{c_2} & -P_{c_2} & 0 & -P_{ag} \\
0 & -P_{c_2} & P_{ag} + P_{c_2} + P_{c_3} & -P_{c_3} & -P_{ag} \\
0 & 0 & -P_{c_3} & P_{ag} + P_{c_3} & -P_{ag} \\
0 & -P_{ag} & -P_{ag} & -P_{ag} & 3P_{ag} + P_s
\end{bmatrix}
\]

\[
A_{p_4} = \begin{bmatrix}
3P_{ag} + P_s & -P_{ag} & -P_{ag} & -P_{ag} & 0 \\
-P_{ag} & P_{ag} + P_{c_3} & -P_{c_3} & 0 & 0 \\
-P_{ag} & -P_{c_3} & P_{ag} + P_{c_2} + P_{c_3} & -P_{c_2} & 0 \\
-P_{ag} & 0 & -P_{c_2} & P_{ag} + P_{c_1} + P_{c_2} & -P_{c_1} \\
0 & 0 & 0 & -P_{c_1} & P_m + P_{c_1}
\end{bmatrix}
\]

Not in this case
Challenge of Mesh-Based Implementation

- Physical movement between magnetic materials
  - Permeances go to zero with non-overlap
  - Reluctances go to infinity with non-overlap
    - Loops are position dependent

- Not a challenge for stationary magnetics
  - Can use discrete rotor positions for machine design and create set of stationary magnetics

- Recent research has shown promising algorithmic method to automate loop changes
Conclusions

- Nodal-based and Mesh-based MEC have different numerical properties

- Mesh-based has better convergence in Newton Raphson formulations
  - ‘Exact’ nodal formulation of Jacobian highly ill-conditioned
  - Approximate Jacobian better conditioned, but still poor relative to mesh-based

- In cases where motion represented, Mesh-based formulation must deal with infinite reluctance

- Algorithmic means of dealing with infinite reluctance is being considered